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ON THE ITERATION OF POWER SERIES IN TWO VARIABLES

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§1. Introduction

It was shown by Koenigs, [5], that if

$$f(x) = \sum_{k=1}^{\infty} a_k x^k \tag{1}$$

has a non-zero radius of convergence and $|a_1|$ is less than 1 and not equal to zero, then for $|x|$ sufficiently small, there exists a function of two variables, $f(x, t)$ possessing the properties of a generalized iterate, which is to say

$$\begin{aligned} & f(x, 0) = x, \\ & f(x, n) = \text{n-th iterate of } x, \text{ for } n \text{ a positive integer}, \\ & f\big(f(x, s), t\big) = f(x, s+t), \quad \text{for } s, t \geq 0 . \end{aligned} \tag{2}$$

This function has the elegant representation,

$$f(x, t) = \phi^{-1}\left[a_1^t \phi(x)\right], \tag{3}$$

where

$$\phi(x) = \lim_{n \to \infty} f^{(n)}(x) / a_1^n . \tag{4}$$

The function $\phi(x)$ is itself an interesting and important function, since it 'linearizes' $f(x)$, i.e.,

$$\phi\big(f(x)\big) = a_1 \phi(x) . \tag{5}$$

The problem of finding a $\phi(x)$ satisfying (5), given $f(x)$, has a long history, going back to Abel. For further discussion and references, we refer to Hadamard, [4].

Let us now turn our attention to power series in two variables. Let

$$(6)\qquad f(x, y) = \sum a_{k\ell}x^k y^\ell, \qquad k, \ell \geq 0, \quad k + \ell \geq 1$$

$$g(x, y) = \sum b_{k\ell}x^k y^\ell,$$

be convergent for $|x|$ and $|y|$ sufficiently small and let us make the assumption that the characteristic roots of the matrix of the coefficients of the linear terms,

$$(7)\qquad A = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix},$$

are non-zero and less than unity in absolute value.

The iterates of the function-pair (f, g) are defined as follows:

$$(8)\qquad f_n(x, y) = f_{n-1}(f, g)$$

$$g_n(x, y) = g_{n-1}(f, g), \qquad n = 1, 2, \cdots,$$

where we set $f_0(x, y) = x$, $g_0(x, y) = y$.

It is natural to ask whether we can find continuous iterates, i.e., a pair of functions of three variables, $\bigl(f(x, y, t), g(x, y, t)\bigr)$, for which

$$(9)\qquad f(x, y, n) = f_n(x, y)$$

$$g(x, y, n) = g_n(x, y), \qquad n = 0, 1, 2, \cdots,$$

and

$$(10)\qquad f\bigl(f(x, y, s), g(x, y, s), t\bigr) = f(x, y, s + t),$$

$$g\bigl(f(x, y, s), g(x, y, s), t\bigr) = g(x, y, s + t),$$

for s and $t \geq 0$. In general, these functions will be defined only for $|x|$ and

$|y|$ sufficiently small and $s, t \geq 0$.

Following Koenigs' idea, which is really an application of the method of successive approximations, the existence of these functions, together with an explicit representation, will follow readily if we establish the existence of functions, $\phi(x, y)$, $\psi(x, y)$, satisfying the analogue of (5), namely, the vector-matrix equation

$$(11) \qquad \begin{pmatrix} \phi(f, g) \\ \psi(f, g) \end{pmatrix} = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix} \begin{pmatrix} \phi(x, y) \\ \psi(x, y) \end{pmatrix} .$$

Before proceeding any further, let us note that by means of some elementary transformations, we may reduce the problem to the case where f and g have the simpler forms,

$$(12) \qquad f = \rho x + \sum_{k+\ell \geq 2} a_{k\ell} x^k y^\ell$$

$$g = \sigma y + \sum_{k+\ell \geq 2} b_{k\ell} x^k y^\ell , \qquad \rho, \sigma \neq 0,$$

provided that A does not have multiple characteristic roots. If A has multiple characteristic roots a representation similar to that given above holds and the details are very much the same.

Under the assumption that A has distinct characteristic roots, the equations in (11) take the simpler form

$$(13) \qquad \phi(f, g) = \rho\phi(x, y),$$

$$\psi(f, g) = \sigma\psi(x, y).$$

By analogy with (4), we might expect that

$$(14) \qquad \phi = \lim_{n \to \infty} f_n(x, y) / \rho^n$$

$$\psi = \lim_{n \to \infty} g_n(x, y) / \sigma^n .$$

Unfortunately, very simple examples show that these limits may not exist. For example, consider

$$(15) \qquad f = \rho x$$

$$g = \sigma y + x^2 .$$

It is easily shown by induction that

$$(16) \qquad f_n = \rho^n x$$

$$g_n = \sigma^n y + (\sigma^{n-1} + \sigma^{n-2}\rho^2 + \cdots + \rho^{2n})x^2 .$$

Thus,

$$(17) \qquad \phi = \lim_{n \to \infty} f^n / \rho^n = x$$

$$\psi = \lim_{n \to \infty} g^n / \sigma^n = y + \frac{x^2}{\sigma} \lim_{n \to \infty} \left(1 + \frac{\rho^2}{\sigma} + \cdots + \left(\frac{\rho^2}{\sigma}\right)^n\right).$$

This last limit exists only for $|\rho^2/\sigma| < 1$.

Despite the fact that a direct application of Koenigs' method fails, proofs of the existence of functions satisfying (13) have been given, provided that a certain condition is satisfied by ρ and σ which we shall discuss below, using the method of successive approximations, or the method of majorants, the last being particularly suited to the analytic case which we are treating here, cf. Leau, [6], Grevy, [3].

If we employ the method of undetermined coefficients to find functions ϕ and ψ satisfying (13) for the functions f and g defined by (15), we find without difficulty,

$$(18) \qquad \phi = x$$

$$\psi = y + \frac{x^2}{\sigma - \rho^2},$$

provided that $\sigma \neq \rho^2$.

If we compare the solution of (18) with the attempted solution of (17), we are at once struck by the fact that $1/(\sigma - \rho^2)$ is the generalized sum of $\sum_{n=0}^{\infty} (\rho^2/\sigma)^n/\sigma$, using an appropriate summability method, or, alternatively, $1/(1 - z)$ is the analytic continuation of the function defined by $\sum_{n=0}^{\infty} z^n$ for $|z| < 1$.

This fact leads us to conjecture that a suitable interpretation of the limits in (17) will yield the required functions ϕ and ψ, and we shall show below that this is actuallyso. Although the method we employ is equally applicable to the general case of power series in n variables, for the sake of simplicity we restrict ourselves to the two-variable case.

We feel that this method of establishing the existence of the linearizing functions ϕ and ψ is of interest because it furnishes another example of the meta-mathematical axiom that formal methods correctly interpreted yield correct results. An example was given in a previous paper, [1], where it was shown that the Liouville-Neumann solution

$$(19) \qquad f = g + \lambda \int_0^1 K(x, y) g(y) dy + \cdots$$

of the Fredholm integral equation

$$(20) \qquad f = g(x) + \lambda \int_0^1 K(x, y) f(y) dy$$

is summable by an appropriate summability method to the Fredholm solution whenever λ is not equal to a characteristic value.

In many cases, these formal methods, combined with summability methods, are useful for computational purposes, cf. Buchner, [2].

§2. Some Intermediary Functions

Let us set

$$(1) \qquad f = \rho x(1 + u) + r(y)$$

$$g = \sigma y(1 + v) + s(x),$$

where u and v are power series in x and y lacking constant terms and r and s are power series in y and x, respectively, lacking constant and first degree terms. We shall employ the following notation for the iterates:

$$(2) \qquad f_{n+1} = f(f_n, g_n), \; g_{n+1} = g(f_n, g_n),$$

$$s_{n+1} = s(f_n), \qquad r_{n+1} = r(g_n),$$

$$u_{n+1} = u(f_n, g_n), \; v_{n+1} = v(f_n, g_n), \qquad n = 1, 2, \cdots,$$

where $f_1 = f$, $g_1 = g$. Using this notation, we have, for example,

$$(3) \qquad f_2 = \rho^2 x(1 + u_1)(1 + u_2) + \rho r_1(1 + u_2) + r_2,$$

$$g_2 = \sigma^2 y(1 + v_1)(1 + v_2) + \sigma s_1(1 + v_2) + s_2,$$

and it is not difficult to establish inductively that

(4) $$f_n = \rho^n x \prod_{k=1}^{n} (1 + u_k) + \rho^{n-1} r_1 \prod_{k=2}^{n} (1 + u_k) + \cdots + r_n ,$$

or

(5) $$\frac{f_n}{\rho^n} = \prod_{k=1}^{n} (1 + u_k) \left[x + \sum_{\ell=1}^{n} \frac{r_\ell}{\rho^\ell \prod_{k=1}^{\ell} (1 + u_k)} \right] ,$$

and, similarly,

(6) $$\frac{g_n}{\sigma^n} = \prod_{k=1}^{\infty} (1 + v_k) \left[y + \sum_{\ell=1}^{n} \frac{s_\ell}{\sigma^\ell \prod_{k=1}^{\ell} (1 + v_k)} \right] .$$

If the infinite products converge, and also the infinite series, we may pass to the limit as $n \to \infty$ and obtain ϕ and ψ. It is easy to demonstrate, after the fashion of Koenigs, that for $|x|$ and $|y|$ sufficiently small, the infinite products converge, on the assumption that $|\rho|$, $|\sigma|$ are both less than one. We have already assumed that $\rho \neq \sigma$, and without loss of generality we may take $|\rho| \geq |\sigma|$. Actually, in place of $\rho \neq \sigma$, one need only assume that the matrix A is diagonalizable.

If $|\rho| \geq |\sigma| > |\rho|^2$, the infinite series which occur in the limiting forms of (5) and (6) converge. If $\sigma = \rho^n$, $n = 2, 3, \cdots$, the functions ϕ and ψ need not exist, as we see by taking

(7) $$f = \rho x$$

$$g = \sigma y + \sum_{k=2}^{\infty} c_k x^k ,$$

for which ϕ and ψ are given by

$$(8) \qquad \phi = x$$

$$\psi = y + \sum_{k=2}^{\infty} \frac{c_k x^k}{\rho^k - \sigma} .$$

To shorten our proof, we shall assume $\sigma \neq \rho^2$, $|\rho|^2 \geq |\sigma| > |\rho|^3$, which is the first interesting case.

Let us define

$$(9) \qquad F(z, x, y) = \sum_{n=1}^{\infty} \frac{r_n}{\prod_{k=1}^{n} (1 + u_k)} z^n,$$

$$G(z, x, y) = \sum_{n=1}^{\infty} \frac{s_n z^n}{\prod_{k=1}^{n} (1 + v_k)} .$$

The convergence of these functions for $|x|$, $|y|$ small and $|z| < 1/|\rho|^2$ follows from inequalities for r_n and s_n we derive below. We also require the functions

$$(10) \qquad \phi(z, x, y) = \prod_{k=1}^{\infty} (1 + u_k)\left[x + F(z, x, y)\right]$$

$$\psi(z, x, y) = \prod_{k=1}^{\infty} (1 + v_k)\left[y + G(z, x, y)\right].$$

<u>Lemma 1</u>. <u>For</u> $|z| < 1/|\rho|^2$,

$$(11) \qquad \phi(z, f, g) = \frac{1}{z} \phi(z, x, y) + x(\rho - \frac{1}{z}) \prod_{k=1}^{\infty} (1 + u_k)$$

$$\psi(z, f, g) = \frac{1}{z} \psi(z, x, y) + y(\sigma - \frac{1}{z}) \prod_{k=1}^{\infty} (1 + v_k) .$$

Proof of Lemma: We have

(12) $$F(z, f, g) = \sum_{n=1}^{\infty} \frac{r_{n+1} z^n}{\prod_{k=2}^{n+1} (1 + u_k)} = \frac{(1 + u_1)}{z} F(z, x, y) - r_1,$$

whence

(13) $$\phi(z, f, g) = \prod_{k=2}^{\infty} (1 + u_k)\left[f + F(z, f, g)\right]$$

$$= \prod_{k=1}^{\infty} (1 + u_k)\left[\frac{f - r_1}{(1 + u_1)} + \frac{1}{z} F(z, x, y)\right]$$

$$= \prod_{k=1}^{\infty} (1 + u_k)\left[\rho x + \frac{1}{z}\left(\frac{\phi}{\prod_{k=1}^{\infty} (1 + u_k)} - x\right)\right]$$

$$= \frac{1}{z} \phi + x(\rho - \frac{1}{z}) \prod_{k=1}^{\infty} (1 + u_k),$$

and similarly for ψ.

From (11), we see that if ψ possesses an analytic continuation which includes the point $z = 1/\sigma$, then we will have

(14) $$\phi(\frac{1}{\rho}, x, y) = \rho\phi(\frac{1}{\rho}, x, y),$$

$$\psi(\frac{1}{\sigma}, x, y) = \sigma\psi(\frac{1}{\sigma}, x, y),$$

and consequently the desired linearizing functions.

In the next section we prove that this analytic continuation exists.

§3. Proof of Analytic Continuation

We wish to demonstrate:

Lemma 2. The functions $\phi(z, x, y)$, $\psi(z, x, y)$, considered as functions of z for $|x|$ and $|y|$ sufficiently small are meromorphic functions whose only possible singularities are at the points $z = \sigma^{-(m+2)}\rho^{-n}$, $m, n \geq 0$ for ϕ, and $z = \rho^{-(n+2)}\sigma^{-m}$, $m, n \geq 0$ for ψ.

The proof depends upon a continued application of the Hadamard multiplication theorem which we state as

Lemma 3. (Hadamard). If, in $|z| < 1$, the singularities of the functions defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, for $|z| < a < 1$, are at α_i, β_j, respectively, $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, \ell$, then the singularities of the function defined by $h(z) = \sum_{n=0}^{\infty} a_n b_n z^n$ for $|z| < a < 1$, are to be found in the region $|z| < 1$ at the points $\alpha_i \beta_j$, if any singularities are present.

We require first the following crude inequalities:

(1)
$$|f_n| \leq \rho_1^n |x|$$
$$|g_n| \leq \sigma_1^n |x|, \qquad \text{for } |y| \leq |x| \leq \delta,$$
$$|f_n| \leq \rho_1^n |y|$$
$$|g_n| \leq \sigma_1^n |y|, \qquad \text{for } |x| \leq |y| \leq \delta,$$

where $\rho_1 = |\rho| + \varepsilon$, $\sigma_1 = |\sigma| + \varepsilon$ and ε can be made arbitrarily small by taking δ sufficiently small. These inequalities follow from

(2)
$$|f| \leq |\rho|\, |x|\,(1 + |u|) + a_1 |y|^2$$
$$\leq |\rho|\, |x| \left[1 + |u| + a_1 |y| (|y| / |x|)^2\right] \leq \rho_1 |x|$$

if $|y| \leq |x| \leq \delta$. Similarly, we obtain

(3) $$|g| \le |\sigma|\,|y|(1+|v|) + b_1|x|^2 \le |\sigma|\,|x|\left[\left|\frac{y}{x}\right|(1+|v|) + b_1|x|\right] \le \sigma_1|x| .$$

From these results, (1) follows by iteration.

As mentioned above, we shall consider only the case where $|\rho|^3 < |\sigma| \le |\rho|^2$. The series for F in (9) of §2, is majorized, because of the convergence of the infinite product by

(4) $$F_1 = c_1 \sum_{n=1}^{\infty} |r_n| z^n ,$$

which since $|r_n| \le c_2|g_n|^2 \le c_2\sigma_1^{2n}|x|$, (assuming $|x| \ge |y|$), is convergent for $|z| < 1/\sigma_1^2$ and hence for $z = 1/\rho$ if ε is small enough. The investigation of $G(z)$ is more difficult. Since $s(x) = e_2x^2 + e_3x^3 + \cdots$, we have $s_n = e_2f_n^2 + O(|f_n|^3) = e_2f_n^2 + O(\rho_1^{3n})$. Hence, $\sum |f_n|^3 z^n$ converges for $|z| < 1/\rho_1^3$ and thus for $z = 1/\sigma$ if ε is small enough. To ascertain the analytic behavior of G at $z = 1/\sigma$, it is sufficient then to consider the series

(5) $$G_1 = \sum_{n=1}^{\infty} f_n^2 z^n \Big/ \prod_{\kappa=1}^{n} (1+u_\kappa) .$$

In view of the Hadamard multiplication theorem, it is sufficient to examine the two series

(6) $$G_2 = \sum_{n=1}^{\infty} \prod_{k=1}^{n} (1+u_k) z^k , \qquad G_3 = \sum_{n=1}^{\infty} f_n z^n \Big/ \prod_{\kappa=1}^{n} (1+u_\kappa) .$$

Using (5) of §2, we obtain for G_3 the simpler series

(7) $$G_3 = \sum_{n=1}^{\infty} (\rho z)^n \left[\sum_{k=1}^{n} r_k \rho^{-\kappa} \Big/ \prod_{\ell=1}^{\kappa} (1+u_\ell)\right]$$

$$= \frac{1}{1-\rho z} \sum_{n=1}^{\infty} r_\kappa z^\kappa \Big/ \prod_{\ell=1}^{\kappa} (1+u_\ell) .$$

It follows that G_3 has a simple pole at $z = 1/\rho$ and no other singularity within $|z| < \frac{1}{\sigma_1^2}$. Turning now to G_2 we have

$$(8) \qquad G_2 = \prod_{k=1}^{\infty}(1 + u_k)\left\{\sum_{n=1}^{\infty} z^n \Big/ \sum_{k=r+1}^{\infty}(1 + u_k)\right\}$$

$$= \prod_{k=1}^{\infty}(1 + u_k)\left\{\sum_{n=1}^{\infty} z^n\left[1 - \sum_{n+1}^{\infty} u_{n+1} + \alpha_n\right]\right\},$$

where $\alpha_n = O\left(\sum_{n+1}^{\infty} |u_n|^2 + \left(\sum_{n+1}^{\infty} |u_n|\right)^2\right) = O(\rho_1^{2n})$. Hence

$$(9) \qquad G_2 = \frac{\prod_{k=1}^{\infty}(1 + u_k)\left(1 - \sum_{1}^{\infty} u_k\right)}{1 - z} + \frac{\prod_{k=1}^{\infty}(1 + u_k)}{1 - z} \sum_{n=1}^{\infty} u_n z^n + G_5(z),$$

where G_5 has no singularity within $|z| < 1/\rho_1$. Hence, we see that G_2 has a singularity at $z = 1$ and no other within $1/\rho_1$. From the representation

$$(10) \qquad G_1 = \sum_{n=1}^{\infty}\left(\frac{f_n}{\prod_{k=1}^{n}(1 + u_k)}\right)^2 \prod_{k=1}^{n}(1 + u_k) z^n,$$

and Lemma 3, it follows that G_1 has a possible singularity at $z = 1/\rho^2$ and no other within $1/\rho_1^3$. That this singularity at $z = 1/\rho^2$ follows from the fact that the radius of convergence is $z = 1/\rho^2$.

The meromorphic behavior over the whole plane is obtained similarly, using induction.

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BIBLIOGRAPHY

[1] R. Bellman, A note on the summability of formal solutions of linear integral equations, Duke Mathematical Journal, Vol. 17 (1950), pp. 53-55.

[2] Hans Büchner, A special method of successive approximations for Fredholm integral equations, Duke Mathematical Journal, Vol. 15 (1948), pp. 197-206.

[3] P. Grevy, Sur les équations fonctionnelles, Annales de l'Ecole Normale, (1894).

[4] J. Hadamard, Two works on iteration and related questions, Bulletin Amer. Math. Soc., Vol. 50, (1944), pp. 67-75.

[5] G. Koenigs, Recherches sur les integrales de certaines équations fonctionnelles, Ann. Sci. de l'Ecole Norm. Sup. (3) I Suppl. (1884), pp. 3-41. Nouvelles recherches sur les equation fonctionnelles, ibid 2, (1885), pp. 385-404.

[6] L. Leau, Etude sur les équations fonctionnelles à une ou à plusieurs variables, Annales de Toulouse, Vol. XI, (1897).